Analysis IV

(for EL, GM, MX)

Week 11 ~ 05.05.25

Last time: We used the Laplace transform to solve initial value problems for linear ODEs with constant coefficients.

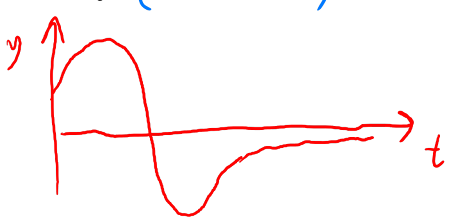
Damped harmonic oscillator:

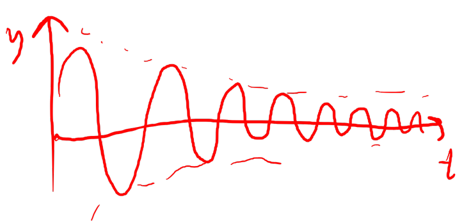
$$\begin{cases} y''(t) + 2a\,y'(t) + w^2\,y(t) = f(t), & t>0 \\ y(0) = y_0, \quad y'(0) = v_0 \end{cases}$$

$$\boxed{a>0,\ w\neq 0}$$

We computed the explicit solutions in the cases

$w^2 = a^2$ (critically damped) and $w^2 > a^2$ (under-damped)





Case III: $w^2 < a^2$ (over-damped)

Use the fact that

$$\mathcal{L}[\cosh(\beta t)](z) = \frac{z}{z^2-\beta^2}, \quad \mathcal{L}[\sinh(\beta t)](z) = \frac{\beta}{z^2-\beta^2}$$

(Exercise, but can also be derived from the formulas for $\cos(\cdot)$, $\sin(\cdot)$ using the fact that $\cosh(x) = \cos(ix)$, $\sinh(x) = -i\sin(ix)$)

So: If $\beta = \sqrt{\underbrace{a^2-\omega^2}_{>0}}$, then:

$$\frac{1}{(z+a)^2+\underbrace{\omega^2-a^2}_{=-\beta^2}} = \frac{1}{\beta}\cdot\frac{\beta}{(z+a)^2-\beta^2} = \frac{1}{\beta}\mathcal{L}[e^{-at}\cdot\sinh(\beta t)](z)$$

and

$$\frac{z+a}{(z+a)^2+\underbrace{\omega^2-a^2}_{=-\beta^2}} = \mathcal{L}[e^{-at}\cdot\cosh(\beta t)](z)$$

So, from our expression for $Y(z) = \mathcal{L}[y](z)$:

$$Y(z) = \frac{1}{(z+a)^2+\omega^2-a^2}F(z) + \frac{z+a}{(z+a)^2+\omega^2-a^2}\cdot y_0 + \frac{1}{(z+a)^2+\omega^2-a^2}\cdot(ay_0+v_0)$$

$$Y(z) = \mathcal{L}\left[\frac{1}{\sqrt{a^2-\omega^2}} e^{-at} \sinh(\sqrt{a^2-\omega^2}\, t)\right](z) \cdot F(z)$$

$$+ \mathcal{L}\left[e^{-at} \cdot \cosh(\sqrt{a^2-\omega^2}\, t)\right](z) \cdot y_0$$

$$+ \mathcal{L}\left[\frac{1}{\sqrt{a^2-\omega^2}} \cdot e^{-at} \sinh(\sqrt{a^2-\omega^2}\, t)\right](z) \cdot (ay_0 + v_0)$$

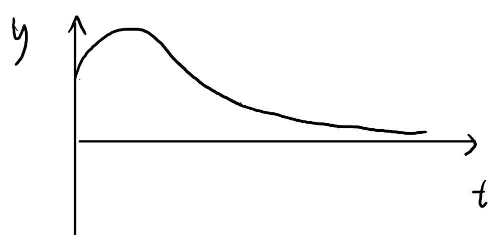
So:

$$y(t) = \int_0^t \frac{1}{\sqrt{a^2-\omega^2}} \cdot e^{-a(t-s)} \sinh(\sqrt{a^2-\omega^2}(t-s)) \cdot f(s)\, ds$$

$$+ e^{-at} \cdot \cosh(\sqrt{a^2-\omega^2}\, t) \cdot y_0$$

$$+ \frac{1}{\sqrt{a^2-\omega^2}} \cdot e^{-at} \cdot \sinh(\sqrt{a^2-\omega^2}\, t) \cdot (ay_0 + v_0)$$

When $f = 0$:



(similar to the critically damped case)

Remarks:

1) In all cases: When $y_0 = 0$, $v_0 = 0$ (so only the forcing affects the solution):

$$y(t) = \int_0^t G(t-s) f(s)\, ds = G * f(t)$$

$G(t)$: Solution in the case when $f=0$, $y_0 = 0$, $v_0 = 1$

(fundamental solution)

2) <u>Resonance</u>:

In the case without damping $(a=0)$.

When $y_0 = 0$, $v_0 = 0$:

$$y(t) = \int_0^t \frac{1}{|w|} \sin(|w|(t-s)) \cdot f(s)\, ds$$

Suppose that $f(s) = \sin(\gamma \cdot s)$ for some $\gamma \in \mathbb{R}$:

- If $\gamma \neq \pm \omega$: $\sup_t |y(t)| < +\infty$

- If $\gamma = \pm \omega$: $|y(t)| \longrightarrow +\infty$ as $t \to +\infty$ (resonant forcing).

For the initial value problems we considered so far:
We performed a Laplace transform, found an expression for $\mathcal{L}[y] = Y$ and then guessed an expression for y.
This is fine, since we proved that those initial value problems have a unique solution.
In general, can we determine y uniquely if we know $\mathcal{L}[y](z)$?

Inverse Laplace transform

Recall: If γ_0 is an abscissa of convergence for f (this means, in practice, that asymptotically, as $t \to +\infty$, f behaves like $\sim e^{\gamma_0 t}$ or "polynomial" corrections to it), then $\mathcal{L}[f](z)$ is holomorphic (and, thus, has no singularities) on

$$\{z : \operatorname{Re}(z) > \gamma_0\}$$

In fact: In some sense, the "opposite" is also true: If $\mathcal{L}[f](z)$ is holomorphic in $\{z : Re(z) > \gamma_0\}$ and doesn't "grow" as $|z| \longrightarrow +\infty$, then γ_0 is an abscissa of convergence for f.

Theorem

Suppose that $f : [0, +\infty) \longrightarrow \mathbb{C}$ is a piecewise continuous function and, for some $\gamma \in \mathbb{R}$,

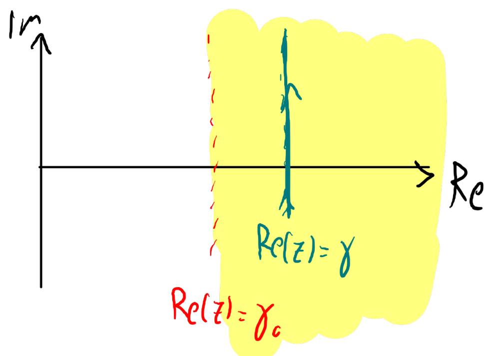
$F(z) = \mathcal{L}[f](z)$ is holomorphic on the region $\{z : Re(z) \geqslant \gamma\}$. and

$$\lim_{T \to +\infty} \int_{-T}^{T} F(\gamma + is) \cdot e^{(\gamma + is)t} \, ds \quad \text{exists for all } t \geqslant 0.$$

(both conditions are automatically true if some $\gamma_0 < \gamma$ is an abscissa of convergence for f)

Then

$$\boxed{\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\gamma+is)\cdot e^{(\gamma+is)t}\,ds \\ &= \frac{1}{2\pi i}\int_{\gamma-i\cdot\infty}^{\gamma+i\cdot\infty} F(z)\cdot e^{zt}\,dz \end{aligned}}$$

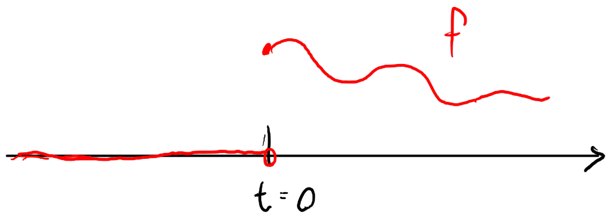


Note: The above integral might not converge absolutely (e.g if $f(t)=1$ for $t\geqslant 0$, $F(z)=\frac{1}{z}$). So we just mean

$$\frac{1}{2\pi}\cdot\lim_{T\to+\infty}\int_{-T}^{+T} F(\gamma+is)\,e^{(\gamma+is)t}\,dt.$$

Proof:

Extend $f : [0, +\infty) \to \mathbb{C}$ to a function on $(-\infty, +\infty)$ by the condition that $f(t) = 0$ for $t < 0$.



Let $g(t) = f(t) \cdot e^{-\gamma t}$.

Take the Fourier transform:

$$\hat{g}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) \cdot e^{-iat} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cdot e^{-(\gamma + ia)t} \, dt$$

$f(t)=0$ for $t \le 0$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} f(t) \cdot e^{-(\gamma + ia)t} \, dt$$

← Laplace transform

$$= \frac{1}{\sqrt{2\pi}} F(\gamma + ia)$$

So by inverse Fourier transform formula:

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{g}(a) \cdot e^{iat} \, da =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\gamma + ia) \cdot e^{iat} \, da$$

$$\xRightarrow{g(t) = f(t) \cdot e^{-\gamma t}} \quad f(t) = g(t) \cdot e^{\gamma t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\gamma + ia) \cdot e^{(\gamma + ia)t} \, dt$$

$\blacksquare$

Remarks:

1) F uniquely determined by $\mathcal{L}[f]$ by this formula (Lerch's theorem)

2) The condition on $\lim_{T \to +\infty} \int_{-T}^{T} F(\gamma + is) \cdot e^{(\gamma + is)t} \, ds$:

Automatically true if $\int_{-\infty}^{+\infty} |F(\gamma + is)| \, ds < +\infty$.

Example:

Let $F(z) = \dfrac{1}{(z+1)(z+2)^2}$.

Find $f : [0, +\infty) \to \mathbb{C}$ such that $\mathcal{L}[f] = F$.

Two ways to do it:

1) Note that

$$\frac{1}{(z+1)(z+2)^2} = \underbrace{\frac{1}{z+1}}_{= \mathcal{L}[e^{-t}]} - \underbrace{\frac{1}{z+2}}_{= \mathcal{L}[e^{-2t}]} - \underbrace{\frac{1}{(z+2)^2}}_{= \mathcal{L}[t \cdot e^{-2t}]}$$

So $f(t) = e^{-t} - e^{-2t} - t \cdot e^{-2t}$

(by uniqueness of Laplace transforms: That's enough)

Usually this is the fastest way.

2) Using the formula for the inverse Laplace transform (combined with the residue theorem)

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{F(\gamma+is)\, e^{(\gamma+is)t}}_{h(z) = F(z)\cdot e^{zt}} \, ds$$

Poles for h: $z = -1, -2$

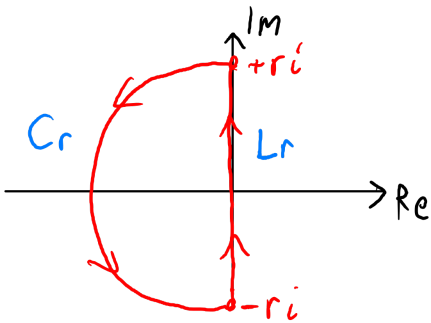
So I can choose $\gamma = 0$:

- F holomorphic on $\{z : Re(z) > -1\}$ which contains $\{Re(z) \geq 0\}$.
- $\int_{-\infty}^{+\infty} |F(\gamma+is)|\, ds < +\infty$.

Inverse Laplace transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(\gamma + is)\, ds$$

$$= \frac{1}{2\pi i} \int_{-\infty i}^{+\infty i} h(z)\, dz$$



Curves: $L_r : [-r, r] \to \mathbb{C}, \quad s \to si$

$C_r : \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{C}, \quad \theta \to r \cdot e^{i\theta}$

$\Gamma = L_r \cup C_r$ (closed, counter clockwise oriented)

When $r > 2$: Γ encloses all the poles of h (namely $z = -1, -2$).

Applying the residue theorem:

$$\int_{\Gamma} h(z)\,dz = 2\pi i \left(\operatorname{Res}_{z=-1}(h) + \operatorname{Res}_{z=-2}(h) \right).$$

Since $h(z) = \dfrac{e^{zt}}{(z+1)(z+2)^2}$:

- $z=-1$: Simple pole

$$\operatorname{Res}_{z=-1}(h) = \lim_{z \to -1} \left((z+1)\, h(z) \right)$$

$$= \lim_{z \to -1} \left(\frac{e^{zt}}{(z+2)^2} \right) = e^{-t}$$

- $z=-2$: Pole of order 2:

$$\operatorname{Res}_{z=-2}(h) = \lim_{z \to -2} \left(\frac{d}{dz} \left((z+2)^2 h(z) \right) \right)$$

$$= \lim_{z \to -2} \left(\frac{d}{dz} \left(\frac{e^{zt}}{z+1} \right) \right)$$

$$= -e^{-2t}(t+1)$$

So: $$\int_{\Gamma} h(z)\,dz = 2\pi i\left(e^{t} - e^{-2t} - t\cdot e^{-2t}\right)$$

Now:

$$\underbrace{\frac{1}{2\pi i}\int_{\Gamma} h(z)\,dz}_{e^{-t} - e^{-2t} - t\cdot e^{-2t}} = \underbrace{\frac{1}{2\pi i}\int_{L_r} h(z)\,dz}_{\xrightarrow{r\to+\infty} f(t)} + \frac{1}{2\pi i}\int_{C_r} h(z)\,dz$$

So: it suffices to show that

$$\left|\int_{C_r} h(z)\,dz\right| \xrightarrow{r\to+\infty} 0$$

Indeed:

$$\left|\int_{C_r} h(z)\,dz\right| = \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(r\cdot e^{i\theta})\cdot i r e^{i\theta}\,d\theta\right|$$

$$|i\cdot r\cdot e^{i\theta}| = r$$

$$\leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |h(r\cdot e^{i\theta})| \cdot r \cdot d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{|e^{r\cdot e^{i\theta} t}|}{|r\cdot e^{i\theta}+1| \cdot |r\cdot e^{i\theta}+2|^2} \cdot r \, d\theta \qquad \mathrm{I}$$

- $$|e^{r\cdot e^{i\theta} t}| = |e^{r(\cos\theta + i\sin\theta)t}|$$
$$= |e^{r\cdot\cos\theta\cdot t} \cdot e^{i r \sin\theta t}|$$
$$= e^{r\cdot\cos\theta\cdot t} \leq 1$$

Since $r > 0$, $\cos\theta \leq 0$ $\left(\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right)$, $t \geq 0$.

- $$\underbrace{|r\cdot e^{i\theta}|}_{=r} - 1 \leq |r\cdot e^{i\theta}+1| \leq \underbrace{|r\cdot e^{i\theta}|}_{=r}+1$$

and, similarly

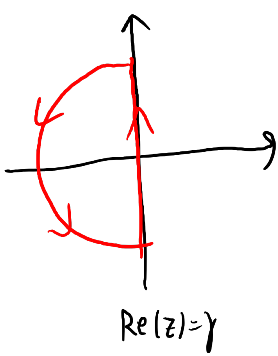
$$r - 2 \leq |r \cdot e^{i\theta} + 2| \leq r + 2$$

So: $$I \leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{(r-1)(r-2)^2} \cdot r \, d\theta$$

$$\longrightarrow 0 \quad \text{as } r \to +\infty$$

$\blacksquare$

Remark on the choice of the closed loop Γ:



We chose the half-circle to be on the left (for $Re(z) < \gamma$) because in that region, $|e^{zt}| \leq e^{\gamma t}$

(so this term remains under control when $r \to +\infty$). This choice always helps when computing inverse Laplace transforms in this way.

Applications to partial differential equations

We will apply the theory of Fourier/Laplace transforms to solve initial-boundary value problems for linear partial differential equations with constant coefficients.

E.g: heat equation:

$$\begin{cases} \partial_t u(t,x) - \partial^2_{xx} u(t,x) = f(t,x), & 0<x<L, \\ & t>0 \\ u(0,x) = u_0(x) & \\ u(t,0) = 0, \quad u(t,L) = 0 & \end{cases}$$

Evolution of temperature in a medium (interval) with fixed temperature at the endpoints $x=0, L$ and heat source f.

General procedure:

- Laplace transform in semi-infinite directions ($t \in [0,+\infty)$)
- Fourier transform in doubly infinite directions ($t \in (-\infty,+\infty)$)
- Fourier series decomposition in bounded intervals ($x \in (0,L)$).